Proof of the Somos-4 Hankel Determinants Conjecture

Guoce Xin

ABSTRACT. By considering the fundamental equation $x = y - y^2 = z - z^3$, Somos conjectured that the Hankel determinants for the generating series $y(z)$ are the Somos-4 numbers. We prove this conjecture by using the quadratic transformation for Hankel determinants of Sulanke and Xin.

1. Introduction

A generating function $Q(x) = \sum_{n \geq 0} q_n x^n$ defines a sequence of Hankel matrices $H_1, H_2, H_3, \ldots$, where H_n is an n by n matrix with entries $(H_n)_{i,j} = q_{i+j-2}$. Hankel determinants are determinants of these matrices. Traditionally, H_0 is defined to be the empty matrix with determinant 1.

In the year of 2000, Somos [6] considered the fundamental equation $x = y - y^2 = z - z^3$. He observed that three types of expansions give nice Hankel determinants. The first one is by expanding y as a series in x, which gives the generating function for Catalan numbers; the second one is by expanding y as a series in z, which gives a generating function related to Catalan and Motzkin numbers; the third one is by expanding z as a series in x, which gives the generating function for ternary trees. The first case was known by Shapiro [5], the third case was proved independently in [1, 2, 8], and the second case, known as the *Somos-4 conjecture*, is still open.

The Somos-4 conjecture can be restated as follows. Expanding y as a series in z gives

$$y = z + z^2 + z^3 + 3z^4 + 8z^5 + 23z^6 + \cdots.$$

Let $Q(z) = (y - z)/z^2$ and let $s_n = \det H_n(Q)$.

CONJECTURE 1 (Somos-4). *The Hankel determinants s_n defined above satisfy the recursion*

$$(1) \qquad s_n s_{n-4} = s_{n-1} s_{n-3} + s_{n-2}^2,$$

with initial conditions $s_0 = 1, s_1 = 1, s_2 = 2, s_3 = 3$.

For instance,

$$H_3(Q) = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 3 & 8 \\ 3 & 8 & 23 \end{pmatrix}, \qquad s_3 = \det H_3(Q) = 3.$$

Our main objective in this paper is to prove the above conjecture.

There are many classical tools of continued fractions for evaluating Hankel determinants, such as the J-fractions in Krattenthaler [4] or Wall [9] and the S-fractions in Jones and Thron [3, Theorem 7.2]. Our tool is by Sulanke and Xin's quadratic transformation for Hankel determinants [7] developed from the continued fraction method of Gessel and Xin [2].

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2. Solving a system of recurrences

Proposition 4.1 of [**7**] defines a quadratic transformation $\mathcal{T}$, and asserts that for certain generating function F, we can find $\mathcal{T}(F)$ such that $\det(H_n(F)) = a \det(H_{n-d-1}(\mathcal{T}(F)))$, where a is a constant and d is a nonnegative integer. See [**7**] for detailed information. Here we only need the following special case.

LEMMA 2. *Suppose $a \neq 0$. If the generating functions $F(x)$ and $G(x)$ are uniquely defined by*

$$F(x) = \frac{a + bx}{1 + cx + dx^2 + x^2(e + fx)F(x)},$$

$$G(x) = \frac{-\frac{a^3 e + a^2 d - acb + b^2}{a^2} - \frac{a^4 f + ca^3 d - c^2 a^2 b + 2\,cab^2 - ba^2 d - b^3}{a^3}x}{1 + cx - \frac{-2\,acb + 2\,b^2 + a^2 d}{a^2}x^2 + x^2(-1 - \frac{b}{a}x)G(x)},$$

then $\det H_n(F) = a^n \det H_{n-1}(G)$.

Our proof is by iterative application the above lemma. To be precise, define $Q_0(x) = Q(x)$, and recursively define $Q_{n+1}(x)$ to be the unique power series solution of

$$(2) \qquad Q_{n+1}(x) = \frac{a_{n+1} + b_{n+1}x}{1 + c_{n+1}x + d_{n+1}x^2 + x^2(e_{n+1} + f_{n+1}x)Q_{n+1}(x)},$$

where

$$(3) \qquad a_{n+1} = -\frac{a_n{}^3 e_n + a_n{}^2 d_n - a_n c_n b_n + b_n{}^2}{a_n{}^2}$$

$$(4) \qquad b_{n+1} = -\frac{a_n{}^4 f_n + c_n a_n{}^3 d_n - c_n{}^2 a_n{}^2 b_n + 2\,c_n a_n b_n{}^2 - b_n a_n{}^2 d_n - b_n{}^3}{a_n{}^3}$$

$$c_{n+1} = c_n$$

$$(5) \qquad d_{n+1} = -\frac{-2\,a_n c_n b_n + 2\,b_n{}^2 + a_n{}^2 d_n}{a_n{}^2}$$

$$e_{n+1} = -1$$

$$(6) \qquad f_{n+1} = -\frac{b_n}{a_n}$$

It is straightforward to represent $Q(x)$ as the unique power series solution of

$$Q(x) = \frac{1 - x}{1 - 2x - x^2 Q(x)}.$$

Therefore we shall set $a_0 = 1, b_0 = -1, c_0 = -2, d_0 = 0, e_0 = -1, f_0 = 0$. By Lemma 2, one can deduce that $\det(H_n(Q)) = a_0^n a_1^{n-1} \cdots a_{n-1}$. This transforms the recursion for s_n to that for a_n as follows:

$$(7) \qquad a_n a_{n-1} a_{n-2} = 1 + 1/a_{n-1}.$$

We remark that the above recursion implies that $s_3 = s_2 + s_1^2$, which holds for the Somos-4 sequence.

It is a surprise that the recursion system can be solved for arbitrary initial condition. For simplicity, we write $c_n = c$ and assume $e_0 = -1$ (otherwise start with Q_1). Our solution can be stated as follows.

THEOREM 3. *Suppose $c_n = c$, $e_n = -1$, and a_n, b_n, d_n, f_n satisfy the recursion (3,4,5,6). Then*

$$(8) \qquad a_{n+2}a_{n+1} + a_{n+1}a_n = 2a_0 a_1 + a_0(f_0 + f_1 + c)(2f_1 + c) - (a_0(f_0 + f_1 + c))^2/a_{n+1}.$$

PROOF. We shall try to write everything in terms of the a's. Using (6), we can replace b_n with $-a_n f_{n+1}$ everywhere. Therefore (3) becomes

$$(9) \qquad d_n = a_n - a_{n+1} - cf_{n+1} - f_{n+1}^2.$$

Substituting (9) into (4) and simplifying gives

$$f_{n+2}a_{n+1} = a_n f_n + ca_n - ca_{n+1} + f_{n+1}a_n - f_{n+1}a_{n+1},$$

which can be written as

$$a_{n+1}(f_{n+2} + f_{n+1} + c) = a_n(f_{n+1} + f_n + c).$$

That is to say

(10) $$a_{n+1}(f_{n+2} + f_{n+1} + c) = a_0(f_1 + f_0 + c).$$

Substituting (9) into (5) and simplifying gives

$$a_n - a_{n+2} = cf_{n+2} + f_{n+2}^2 - (cf_{n+1} + f_{n+1}^2) = (f_{n+2} - f_{n+1})(f_{n+2} + f_{n+1} + c).$$

Applying (10), we obtain

$$a_n a_{n+1} - a_{n+1} a_{n+2} = a_0(f_1 + f_0 + c)(f_{n+2} - f_{n+1}),$$

which leads to

(11) $$a_0 a_1 - a_{n+1} a_{n+2} = a_0(f_1 + f_0 + c)(f_{n+2} - f_1).$$

Combining (10) and (11), we obtain (8). $\qquad\square$

Now we are ready to prove the Somos-4 Conjecture.

PROOF OF THE SOMOS-4 CONJECTURE. Applying Theorem 3 for the case $a_0 = 1, b_0 = -1, c = -2, d_0 = 0, e_0 = -1, f_0 = 0$, we obtain $a_1 = 2$, $f_1 = 1$, and

(12) $$a_{n+2} = 4/a_{n+1} - a_n - 1/a_{n+1}^2.$$

Recall that we have transformed the recursion (1) to (7), which can be written as

$$a_n a_{n-1}^2 a_{n-2} - 1 - a_{n-1} = 0.$$

By applying (12) (with n replaced by $n - 2$) and simplifying, the above equation becomes

$$4a_{n-2}a_{n-1} - a_{n-2} - a_{n-2}^2 a_{n-1}^2 - 1 - a_{n-1} = 0.$$

Denote by $T(n)$ the left-hand side of the above equation. We claim that $T(n) = 0$ for all n, so that (7) holds and the conjecture follows.

We prove the claim by induction on n. The claim is easily checked to be true for $n = 2$. Assume the claim hold for $n - 1$. By applying (12) (with n replaced by $n - 3$) and simplifying, we obtain

$$T(n) = 4a_{n-3}a_{n-2} - a_{n-2} - a_{n-3} - a_{n-3}^2 a_{n-2}^2 - 1 = T(n - 1) = 0.$$

Thus the claim follows. $\qquad\square$

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CENTER FOR COMBINATORICS, LPMC-TJKLC, NANKAI UNIVERSITY, TIANJIN 300071, P. R. CHINA
E-mail address: gxin@nankai.edu.cn